REPRESENTATIONS AND PROPERTIES OF GENERALIZED A_r STATISTICS, COHERENT STATES AND ROBERTSON UNCERTAINTY RELATIONS

M. Daoud[1]

Faculté des Sciences, Département de Physique, LPMC,
Agadir, Morocco

Abstract

The generalization of A_r statistics, including bosonic and fermionic sectors, is performed by means of the so-called Jacobson generators. The corresponding Fock spaces are constructed. The Bargmann representations are also considered. For the bosonic A_r statistics, two inequivalent Bargmann realizations are developed. The first (resp. second) realization induces, in a natural way, coherent states recognized as Gazeau-Klauder (resp. Klauder-Perelomov) ones. In the fermionic case, the Bargamnn realization leads to the Klauder-Perelomov coherent states. For each considered realization, the inner product of two analytic functions is defined in respect to a measure explicitly computed. The Jacobson generators are realized as differential operators. It is shown that the obtained coherent states minimize the Robertson-Schrödinger uncertainty relation.

[1] m_daoud@hotmail.com

1 Introduction and motivations

The quantum statistics, different from Bose and Fermi ones, have attracted due attention in the literature [1-13] and various versions were formulated. For example, in two space dimensions, one can have a one parameter family of statistics (anyons) interpolating between bosons and fermions [4]. On the other hand, in three and higher space dimensions the parastatistics, developed by Green [1], constitute the natural extension of the usual Fermi and Bose statistics. The interest on these exotic statistics is mainly motivated by their promising applications in the theories of fractional quantum Hall effect [7-8], anyonic super-conductivity [9] and black hole statistics [10]. In the Green generalization of conventional Bose and Fermi statistics, the paraboson or parafermion algebra is generated by r pairs of creation and annihilation operators $(A_i^+ \, , A_i^-)$ $(i = 1, 2, \dots, r)$ satisfying the trilinear relations (which replace the standard bilinear commutation or anti-commutation relations)

$$[[A_i^+, A_j^-]_\pm, A_k^-]] = -2\delta_{ik}A_j^-, \quad [[A_i^+, A_j^+]_\pm, A_k^-]] = -2\delta_{ik}A_j^+ \mp 2\delta_{jk}A_i^+, \quad [[A_i^-, A_j^-]_\pm, A_k^-]] = 0$$

where as usual $[x, y]_\pm = xy \pm yx$ and the sign $+$ (resp. $-$) refer to parabosons (resp. parafermions). It is interesting to mention that the para-Fermi relations are associated with the orthogonal Lie algebra $so(2r+1) = B_r$ [14] and the para-Bose statistics are connected to the orthosymplectic superalgebra $osp(1/2r) = B(0,r)$ [15]. Recently, in view of this connection between Lie algebras and super-algebras, a classification of generalized quantum statistics were derived in the framework of the classical Lie algebras A_r, B_r, C_r and D_r [6,16,17].

In this context, we shall be interested, in the present paper, in generalized class of statistics associated with the classical Lie algebra A_r. The general class of these statistics is defined with the help of the notion of Lie triple systems and the so-called Jacobson operators [18]. The latter operators are known to be closely related to the description, initiated by N. Jacobson, of Lie algebras by a minimal set of generators and relations instead of to the well known Chevally description. The second facet of this work concerns the Bargmann representation associated with generalized A_r statistics. The latter is frequently important in the analysis of quantum field theoretic systems and in connection with path integral methods. Coherent states for A_r statistics system emerges naturally in the Bargmann realization. Coherent states for systems obeying unconventional statistics were extensively investigated in recent years. One may quote coherent states associated with statistics developed in the context of quantum algebras like q-bosons [19] and k-fermions [20]. Coherent states for paraparticles were also constructed: Parabose coherent states were proposed in [21] and Parafermi ones are given in [3,22]. All these states appear to be quantum states closest to the classical ones. The strongest qualitative measure of differences in the behavior of quantum and classical properties is expressed by the Schrödinger-Robertson uncertainty principle [23-24] (see also [25-26]). As we are interested in the generalized A_r statistics, it is natural to ask if the sets of coherent states, which emerge in the construction of analytical representations, minimize the Robertson-Shrödinger uncertainty relation. This matter will con-

stitute the last part of this work.

The paper is organized as follows. Generalized quantum statistics is introduced from a set of Jacobson generators (defined in section 2) satisfying certain triple relations. This generalization includes two fundamental sectors. A fermionic one reproducing the A_r statistics introduced in [6]. The second sector is of bosonic type. For each sector, we give the associated Fock space. A Hamiltonian is derived in terms of the Jacobson generators identified with creation and annihilation operators. In section 3, the first analytic realization of the Fock space for the bosonic A_r statistics is performed. This realization generates the so-called Gazeau-Klauder coherent states [27]. The second realization, presented is section 4, leads to the Klauder-Perelomov coherent states [28-29]. We also realize analytically the Fock space related to the fermionic A_r statistics. In this case, we show that the Jacobson generators act on a over-complete set of coherent states similar to Klauder-Perelomov ones labeling the complex projective spaces $\mathbf{CP}^r$. Differential actions of the Jacobson generators for each obtained realization are given. In the last section, we show that the quantum states, realizing analytically the vector states of A_r statistics, minimize the uncertainty principle. In other words, they minimize the Robertson-Schrödinger uncertainty relation. Some concluding remarks close this work.

2 The generalized A_r statistics

In this section, we introduce the definitions of the Jacobson operators and the generalized A_r statistics viewed as Lie triple system. We give the corresponding Fock space and a Hamiltonian describing a quantum system obeying generalized A_r statistics.

2.1 Jacobson generators

To begin, let us recall the definition of Lie triple systems. A vector space with trilinear composition $[x,y,z]$ is called Lie triple system if the following identities are satisfied:

$$[x,x,x]=0,$$

$$[x,y,z]+[y,z,x]+[z,x,y]=0,$$

$$[x,y,[u,v,w]]=[[x,y,u],v,w]+[u,[x,y,v],w]+[u,v,[x,y,w]].$$

According this definition, we will introduce the generalized A_r statistics as Lie triple system. For this end, we consider the set of $2r$ operators x_i^+ and x_i^- $(i=1,2,...,r)$. Inspired by the para-Fermi case [1] and the example of A_r statistics [6,16], these $2r$ operators should satisfy certain conditions and relations. First, the operators x_i^+ are mutually commuting. A similar statement holds for the operators x_i^-. They also satisfy the following triple relations

$$[[x_i^+,x_j^-],x_k^+]]=-\epsilon\delta_{jk}x_i^+-\epsilon\delta_{ij}x_k^+ \qquad (1)$$

$$[[x_i^+,x_j^-],x_k^-]]=\epsilon\delta_{ik}x_j^-+\epsilon\delta_{ij}x_k^- \qquad (2)$$

where $\epsilon \in \mathbf{R} \setminus \{0\}$. The algebra $\mathcal{A}$ (defined by means of the generators $x_i^{\pm}$ and relations (1) and (2)) is closed under the ternary operation $[x, y, z] = [[x, y], z]$ and define a Lie triple system. Note that for $\epsilon = -1$, the algebra $\mathcal{A}$ reduces to one defining the A_r statistics discussed in [6]. The elements $x_i^{\pm}$ are referred to as Jacobson generators which will be identified later with creation and annihilation operators of a quantum system obeying generalized A_r statistics. We redefine the generators of the algebra $\mathcal{A}$ as $a_i^{\pm} = \frac{x_i^{\pm}}{\sqrt{|\epsilon|}}$. The triple relations (1) and (2) may be rewritten as

$$[[a_i^+, a_j^-], a_k^+]] = -s\delta_{jk}a_i^+ - s\delta_{ij}a_k^+, \tag{3}$$

$$[[a_i^+, a_j^-], a_k^-]] = s\delta_{ik}a_j^- + s\delta_{ij}a_k^- \tag{4}$$

where $s = \frac{\epsilon}{|\epsilon|}$ is the sign of the parameter ϵ and $[a_i^+, a_j^+] = [a_i^-, a_j^-] = 0$. This redefinition is more convenient for our investigation, in particular in determining the irreducible representation associated with the algebra $\mathcal{A}$. As we will see in what follows, the sign of the parameter ϵ play an importance in the representation of the algebra $\mathcal{A}$ and consequently, one can obtain different microscopic and macroscopic statistical properties of the quantum system under consideration.

2.2 The Hamiltonian

To characterize a quantum gas obeying the generalized A_r statistics, we have to specify a Hamiltonian for the system. The operators $a_i^{\pm}$ define creation and annihilation operators for a quantum mechanical system, described by an Hamiltonian H, when the Heisenberg equation of motion

$$[H, a_i^{\pm}] = \pm e_i a_i^{\pm} \tag{5}$$

is fulfilled. The quantities e_i are the energies of the modes $i = 1, 2, ..., r$. One can verify that if $|E\rangle$ is an eigenstate with energy E, $a_i^{\pm}|E\rangle$ are eigenvectors of H with energies $E \pm e_i$. In this respect, the operators $a_i^{\pm}$ can be interpreted as ones creating or annihilating particles. To solve the consistency equation (5), we write the Hamiltonian H as

$$H = \sum_{i=1}^{r} e_i h_i \tag{6}$$

which seems to be a simple sum of "free" (non-interacting) Hamiltonians h_i. However, note that, in the quantum system under consideration, the statistical interactions occur and are encoded in the triple commutation relations (3) and (4). Using the structure relations of the algebra $\mathcal{A}$, the solution of the Heisenberg condition (5) is given by

$$h_i = \frac{s}{r+1}\left[(r+1)[a_i^-, a_i^+] - \sum_{j=1}^{r}[a_j^-, a_j^+]\right] + c \tag{7}$$

where the constant c will be defined later such that the ground state (vacuum) of the Hamiltonian H has zero energy.

2.3 Fock representations

We now consider a Hilbertian representation of the algebra $\mathcal{A}$. Let $\mathcal{F}$ be the Hilbert-Fock space on which the generators of $\mathcal{A}$ act. Since, the algebra $\mathcal{A}$ is spanned by r pairs of Jacobson generators, it is natural to assume that the Fock space is given by

$$\mathcal{F} = \oplus_{n=0}^{\infty} \mathcal{H}^n, \tag{8}$$

where $\mathcal{H}^n \equiv \{|n_1, n_2, \cdots, n_r\rangle\ , n_i \in \mathbf{N}, \sum_{i=1}^{r} n_i = n > 0\}$ and $\mathcal{H}^0 \equiv \mathbf{C}$. The action of $a_i^{\pm}$, on $\mathcal{F}$, are defined by

$$a_i^{\pm}|n_1, \cdots, n_i, \cdots, n_r\rangle\ = \sqrt{F_i(n_1, \cdots, n_i \pm 1, \cdots, n_r)}|n_1, \cdots, n_i \pm 1, \cdots, n_r\rangle \tag{9}$$

extended linearly, where the functions F_i are called the structure functions and are to be non-negatives so that all states are well defined. To determine the expressions of the functions F_i in terms of the quantum numbers $n_1, n_2, \cdots, n_r$, let first assume that $a_i^{-}|0, 0, \cdots, 0\rangle\ = 0$ for all $i = 1, 2, \cdots, r$. This implies that the functions F_i satisfy

$$F_i(n_1, \cdots, n_i, \cdots, n_r) = n_i G_i(n_1, \cdots, n_i, \cdots, n_r), \tag{10}$$

in a factorized form where the new functions G_i are defined such that $G_i(n_1, \cdots, n_i = 0, \cdots, n_r) \neq 0$ for $i = 1, 2, \cdots, r$. Furthermore, since the Jacobson operators satisfy the trilinear relations (3) and (4), these functions should be affine in the quantum numbers n_i:

$$G_i(n_1, \cdots, n_i, \cdots, n_r) = k_0 + (k_1 n_1 + k_2 n_2 + \cdots + k_r n_r). \tag{11}$$

Finally, using the relations $[a_i^{+}, a_j^{+}] = 0$ and $[[a_i^{+}, a_i^{-}], a_i^{+}]] = -2s a_i^{+}$, one obtain $k_i = k_j$ and $k_i = s$, respectively. For convenience, we set $k_0 = k - \frac{1+s}{2}$ assumed to be a non-vanishing integer. The actions of the Jacobson generators on the states spanning the Hilbert-Fock space $\mathcal{F}$ are now given by

$$a_i^{-}|n_1, \cdots, n_i, \cdots, n_r\rangle\ = \sqrt{n_i(k_0 + s(n_1 + n_2 + \cdots + n_r))}|n_1, \cdots, n_i - 1, \cdots, n_r\rangle, \tag{12}$$

$$a_i^{+}|n_1, \cdots, n_i, \cdots, n_r\rangle\ = \sqrt{(n_i + 1)(k_0 + s(n_1 + n_2 + \cdots + n_r + 1))}|n_1, \cdots, n_i{+}1, \cdots, n_r\rangle. \tag{13}$$

The dimension of the irreducible representation space $\mathcal{F}$ is determined by the condition:

$$k_0 + s(n_1 + n_2 + \cdots + n_r) > 0. \tag{14}$$

It depends on the sign of the parameter s. It is clear that for $s = 1$, the Fock space $\mathcal{F}$ is infinite dimensional. However, for $s = -1$, there exists a finite number of basis states satisfying the condition $n_1 + n_2 + \cdots + n_r \leq k - 1$. The dimension is given, in this case, by $\frac{(k-1+r)!}{(k-1)!r!}$. This is exactly the dimension of the Fock representation of A_r statistics discussed in [6]. This condition-restriction is closely related to so-called generalized exclusion Pauli principle according to which

no more than $k-1$ particles can be accommodated in the same quantum state. In this sense, for $s=-1$, the generalized A_r quantum statistics give statistics of fermionic behaviour. They will be termed here as fermionic A_r statistics and ones corresponding to $s=1$ will be named bosonic A_r statistics.

Setting $c=\frac{r}{r+1}sk_0$ in (7) and using the equation (6) together with the actions of creation and annihilation operators (12-13), one has

$$H|n_1,\cdots,n_i,\cdots,n_r\rangle \ = \sum_{i=1}^{r} e_i n_i |n_1,\cdots,n_i,\cdots,n_r\rangle. \tag{15}$$

It is remarkable that, for $s=-1$, the spectrum of H is similar (with a slight modification) to energy eigenvalues of the A_r Calogero model (see for instance Eq.(1.2) in [30]). The latter describe the dynamical model containing $r+1$ particles on a line with long rang interactions and provides a microscopic realization of fractional statistics [13,31].

Finally, we point out one interesting property of the generalized A_r statistics. Introduce the operators $b_i^{\pm}=\frac{a_i^{\pm}}{\sqrt{k}}$ for $i=1,2,\cdots,r$ and consider k very large. From equations (12) and (13), we obtain

$$b_i^-|n_1,\cdots,n_i,\cdots,n_r\rangle \ \approx \sqrt{n_i}|n_1,\cdots,n_i-1,\cdots,n_r\rangle, \tag{16}$$

$$b_i^+|n_1,\cdots,n_i,\cdots,n_r\rangle \ \approx \sqrt{n_i+1}|n_1,\cdots,n_i+1,\cdots,n_r\rangle. \tag{17}$$

In this limit, the generalized A_r statistics (fermionic and bosonic ones) coincide with th Bose statistics and the Jacobson operators reduce to Bose ones (creation and annihilation operators of harmonic oscillators).

Besides the Fock representation discussed in this section, it is interesting to look for analytical realizations of the space representation associated with the Fock representations of the generalized A_r statistics. These realizations constitute an useful analytical tool in connection with variational and path integral methods to describe the quantum dynamics of the system described by the Hamiltonian H.

3 Bargmann realization and Gazeau-Klauder coherent states

This section is devoted to a realization à la Bargmann using a suitably defined Hilbert space of entire analytic functions associated with the bosonic A_r statistics introduced above. In this first analytic realization, the Jacobson creation operators are realized as simple multiplication by some complex variables. As by product, this realization generates, in a natural way, the Gazeau-Klauder coherent states associated to a quantum mechanical system described by the Hamiltonian given by (6) and (7) for the particular case $s=1$. To begin, we realize the vectors $|k;n_1,\cdots,n_r\rangle$ as powers of complex variables $\omega_1,\cdots,\omega_r$ on which the Jacobson creations operators a_i^+ act as multiplication by ω_i

$$|k;n_1,\cdots,n_r\rangle \longrightarrow C_{k;n_1,\cdots,n_r}\omega_1^{n_1}\cdots\omega_r^{n_r} \tag{18}$$

where the set of coefficients $C_{k;n_1,\cdots,n_r}$ occurring in the last expression will be determined in what follows. Equation (13) leads to the following recursion relation

$$C_{k;n_1,\cdots,n_i,\cdots,n_r} = ((n_i+1)(k+n_1+\cdots+n_i+\cdots+n_r))^{\frac{1}{2}} C_{k;n_1,\cdots,n_i+1,\cdots,n_r}. \tag{19}$$

Solving this equation, we obtain

$$C_{k;n_1,\cdots,n_i,\cdots,n_r} = \left[\frac{(k-1+n-n_i)!}{n_i!(k-1+n)!}\right]^{\frac{1}{2}} C_{k;n_1,\cdots,0,\cdots,n_r}. \tag{20}$$

where $n = n_1+n_2+\cdots+n_r$. We repeat this procedure for all $i = 1,2,\cdots,r$ and setting $C_{k;0,\cdots,0} = 1$, we obtain

$$C_{k;n_1,\cdots,n_i,\cdots,n_r} = \left[\frac{(k-1)!}{n_1!\cdots n_r!(k-1+n)!}\right]^{\frac{1}{2}}. \tag{21}$$

If we define the operators N_i $(\neq a_i^+ a_i^-)$ such that

$$N_i|k;n_1,\cdots,n_i\cdots,n_r\rangle = n_i|k;n_1,\cdots,n_i\cdots,n_r\rangle, \tag{22}$$

it is easy to see that the operators N_i act in this differential realization as

$$N_i \longrightarrow \omega_i \frac{\partial}{\partial\omega_i}. \tag{23}$$

To define the differential actions of the annihilation operators a_i^-, we use their actions on the Fock space (Eq.12) together with the equation (23). One has

$$a_i^- \longrightarrow k\frac{\partial}{\partial\omega_i} + \omega_i\frac{\partial^2}{\partial^2\omega_i} + \frac{\partial}{\partial\omega_i}\sum_{i\neq j}\omega_j\frac{\partial}{\partial\omega_j}. \tag{24}$$

A general vector

$$|\psi\rangle = \sum_{n_1,\cdots,n_r}\psi_{n_1,\cdots,n_r}|k;n_1,\cdots,n_r\rangle \tag{25}$$

in the Fock space $\mathcal{F}$ now is realized as follows

$$\psi(\omega_1,\cdots,\omega_r) = \sum_{n_1,\cdots,n_r}\psi_{n_1,\cdots,n_r}C_{k;n_1,\cdots,n_r}\omega_1^{n_1}\cdots\omega_r^{n_r}, \tag{26}$$

a.e. We define the inner product in this realization in the following form

$$\langle\psi'|\psi\rangle = \int d^2\omega_1\cdots d^2\omega_r K(k;\omega_1,\cdots,\omega_r)\psi'^{\star}(\omega_1,\cdots,\omega_r)\psi(\omega_1,\cdots,\omega_r) \tag{27}$$

where $d^2\omega_i \equiv dRe\omega_i dIm\omega_i$, where K is to be determined and the integration extends over the entire complex space $\mathbf{C^r}$. To compute the density function K, appearing in the definition of the inner product (27), we choose $|\psi\rangle$(resp. $|\psi'\rangle$) to be the vector $|k;n_1,\cdots,n_r\rangle$ (resp. $|k;n'_1,\cdots,n'_r\rangle$). We also assume that K depends only on $\rho_i = |\omega_i|$ for $i = 1,\cdots,r$. This assumption reflects the isotropic condition used in the moment problems. Then, it is a simple matter of computation to show that the function $K(k;\rho_1,\cdots,\rho_r)$ should satisfy the integral equation

$$(2\pi)^r\int_0^\infty\cdots\int_0^\infty d\rho_1\cdots d\rho_r K(k;\rho_1,\cdots,\rho_r)|\rho_1|^{2n_1+1}\cdots|\rho_r|^{2n_r+1} = \frac{n_1!\cdots n_r!(k-1+n)!}{(k-1)!}. \tag{28}$$

A solution of this equation exists [32] (see a nice proof in [33]) in term of the Bessel function

$$K(k;R) = \frac{2}{\pi^r (k-1)!} R^{k-r} K_{k-r}(2R) \tag{29}$$

where $R^2 = \rho_1^2 + \cdots + \rho_r^2$. Note that the analytic function $\psi(\omega_1, \cdots, \omega_r)$ can be viewed as the inner product of the ket $|\psi\rangle$ with a bra $\langle k; \omega_1^\star, \cdots, \omega_r^\star|$ labeled by the complex conjugate of the variables $\omega_1, \cdots, \omega_r$

$$\psi(\omega_1, \cdots, \omega_r) = \mathcal{N} \langle k; \omega_1^\star, \cdots, \omega_r^\star | \psi \rangle \tag{30}$$

where $\mathcal{N} \equiv \mathcal{N}(|\omega_1|, \cdots, |\omega_r|)$ stands for a normalization constant of the states $|k; \omega_1, \cdots, \omega_r\rangle$ to be adjusted later. As a particular case, if we take $|\psi\rangle = |k; n_1, \cdots, n_r\rangle$, we get

$$\langle k; \omega_1^\star, \cdots, \omega_r^\star | k; n_1, \cdots, n_r \rangle = \mathcal{N}^{-1} C_{k;n_1,\cdots,n_r} \omega_1^{n_1} \cdots \omega_r^{n_r}. \tag{31}$$

The last equation implies

$$|k; \omega_1, \cdots, \omega_r\rangle = \mathcal{N}^{-1} \sum_{n_1=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} \left[\frac{(k-1)!}{n_1! \cdots n_r! (k-1+n)!} \right]^{\frac{1}{2}} \omega_1^{n_1} \cdots \omega_r^{n_r} \tag{32}$$

where the normalization constant $\mathcal{N}$ is

$$\mathcal{N}^2(|\omega_1|, \cdots, |\omega_r|) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} \frac{(k-1)!}{n_1! \cdots n_r! (k-1+n)!} |\omega_1|^{2n_1} \cdots |\omega_r|^{2n_r} \tag{33}$$

The states $|k; \omega_1, \cdots, \omega_r\rangle$ are not orthogonal and constitute an over-complete set with respect to the measure given by (29). It is also interesting to remark that they are eigenvectors of the Jacobson operators a_i^- with the eigenvalue ω_i. In this sense, the states $|k; \omega_1, \cdots, \omega_r\rangle$ can be considered as Gazeau-Klauder coherent states associated with a quantum mechanical system whose Hamiltonian is given by (6) and (7).

4 Bargmann realization and Klauder-Perelomov coherent states

4.1 Bosonic A_r statistics

Here, we shall consider the second analytic realization associated with bosonic A_r statistics. We consider the complex domain $\mathcal{D} = \{(z_1, z_2, \cdots, z_r) : |z_1|^2 + |z_2|^2 + \cdots + |z_r|^2 < 1\}$. The reason for this condition will be clarified in the sequel of this subsection. In this realization, the annihilation operators a_i^- are represented as derivation with respect to the complex variables z_i

$$a_i^- \longrightarrow \frac{\partial}{\partial z_i} \tag{34}$$

and the basis elements of the Fock space are realized as follows

$$|k; n_1, \cdots, n_r\rangle \longrightarrow C_{k;n_1,\cdots,n_r} z_1^{n_1} \cdots z_r^{n_r}. \tag{35}$$

Using the action of the annihilation operators on the Fock space $\mathcal{F}$ and the correspondence (35), one obtain the following recursion formula

$$\sqrt{k-1+n_1+\cdots+n_i+\cdots+n_r} C_{k;n_1,\cdots,n_i-1,\cdots,n_r} = \sqrt{n_i} C_{k;n_1,\cdots,n_i,\cdots,n_r} \tag{36}$$

which can be solved in a similar manner that one used above (Eq.19) and setting $C_{k;0,\cdots,0} = 1$. We have

$$C_{k;n_1,\cdots,n_i,\cdots,n_r} = \left[\frac{(k-1+n)!}{n_1!\cdots n_r!(k-1)!}\right]^{\frac{1}{2}} \tag{37}$$

where $n = n_1 + n_2 + \cdots + n_r$. Having the expression of the coefficients C, one can determine the differential action of the Jacobson creation operators. Indeed, using the actions of the generators a_i^+ on the Fock space and the triple relations (3) and (4) , we show that

$$a_i^+ \longrightarrow kz_i + z_i \sum_{j=1}^{r} z_j \frac{\partial}{\partial z_j}; \tag{38}$$

i.e., the Jacobson generators act as first order linear differential operators. Here also, we realize a general vector of the Fock space $\mathcal{F}$ $(s = 1)$

$$|\phi\rangle = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_r=0}^{\infty} \phi_{n_1,n_2\cdots,n_r}|k; n_1, n_2, \cdots, n_r\rangle$$

as

$$\phi(z_1, z_2\cdots, z_r) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_r=0}^{\infty} \phi_{n_1,n_2\cdots,n_r} C_{k;n_1,n_2\cdots,n_r} z_1^{n_1} z_2^{n_2}\cdots z_r^{n_r}, \tag{39}$$

a.e. The inner product of two functions ϕ and ϕ' is defined now as follows

$$\langle\phi'|\phi\rangle = \int\int\cdots\int d^2z_1 d^2z_2\cdots d^2z_r \Sigma(k; z_1, z_2\cdots, z_r)\phi'^{\star}(z_1, z_2\cdots, z_r)\phi(z_1, z_2\cdots, z_r) \tag{40}$$

where the integration is carried out the complex domain $\mathcal{D}$. The computation of the integration measure Σ, assumed to be isotropic , can be performed by choosing $|\phi\rangle = |k; n_1, n_2, \cdots, n_r\rangle$ and $|\phi'\rangle = |k; n'_1, n'_2, \cdots, n'_r\rangle$. It follows that the measure Σ satisfy the following moment equation

$$\int\int\cdots\int d\varrho_1 d\varrho_2\cdots d\varrho_r \Sigma(k; \varrho_1, \varrho_2, \cdots, \varrho_r)\varrho_1^{2n_1+1}\varrho_2^{2n_2+1}\cdots\varrho_r^{2n_r+1} = \frac{n_1!n_2!\cdots n_r!(k-1)!}{(2\pi)^r(k-1+n)!} \tag{41}$$

where $n = n_1 + n_2 + \cdots + n_r$ and $\varrho_i = |z_i|$. To find the isotropic function satisfying the equation (41), we use following result

$$\int_0^1 t_1^{n_1}dt_1 \int_0^{1-t_1} t_2^{n_2}dt_2 \cdots \int_0^{1-t_1-t_2-\cdots-t_{r-1}} t_r^{n_r}(1 - t_1 - t_2 - \cdots - t_r)^{k-r-1}dt_r$$

$$= \frac{n_1!n_2!\cdots n_r!(k-1)!}{(k-1+n)!(k-r)(k-r+1)\cdots(k-1)} \tag{42}$$

which can be easily verified. The measure is then given by

$$\Sigma(k; \varrho_1, \varrho_2, \cdots, \varrho_r) = \pi^{-r}(k-r)(k-r+1)\cdots(k-1)[1 - (\varrho_1^2 + \varrho_2^2 + \cdots + \varrho_r^2)]^{k-r-1}. \tag{43}$$

One can write the function $\phi(z_1, z_2\cdots, z_r)$ as the product of the state $|\phi\rangle$ with some ket $|k; z_1^*, z_2^*\cdots, z_r^*\rangle$ labeled by the complex conjugate of the variables $z_1, z_2, \cdots, z_r$

$$\phi(z_1, z_2, \cdots, z_r) = \mathcal{N}\langle k; z_1^*, z_2^*, \cdots, z_r^*|\phi\rangle. \tag{44}$$

Taking $|\phi\rangle = |k; n_1, n_2, \cdots, n_r\rangle$, we have

$$|k; z_1, z_2, \cdots, z_r\rangle = \mathcal{N}^{-1} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} \left[\frac{(k-1+n)!}{n_1! \cdots n_r!(k-1)!}\right]^{\frac{1}{2}} z_1^{n_1} z_2^{n_2} \cdots z_r^{n_r}. \tag{45}$$

The expansion (45) converges when $|z_1|^2 + |z_2|^2 + \cdots + |z_r|^2 < 1$. In other words, the complex variables $z_1, z_2, \cdots, z_r$ should be in the complex domain $\mathcal{D}$ defined above. The normalization constant in (45) is given by

$$\mathcal{N} = (1 - |z_1|^2 - |z_2|^2 - \cdots - |z_r|^2)^{\frac{k}{2}}. \tag{46}$$

The states (45) are continuous in the labeling, constitute an over complete set in the respect to the measure given by (43) and then are coherent in the Klauder-Perelomov sense. It comes that the quantum states of bosonic A_r statistics system admit two non-equivalent realizations.

4.2 Fermionic A_r statistics

Now, we construct the analytic realization of the irreducible representation related to fermionic A_r statistics ($s = -1$) characterized by the so-called generalized Pauli principle. First, note that since the Fock space is of finite dimension, the Jacobson creation operators can not be represented as a multiplication by some complex variable. Unlike the bosonic A_r statistics, only one realization can be made in this case. It corresponds to one in which the generators a_i^- act as

$$a_i^- \longrightarrow \frac{\partial}{\partial \zeta_i} \tag{47}$$

in the space of the polynomials of the form $C_{k;n_1,n_2,\cdots,n_r} \zeta_1^{n_1} \zeta_2^{n_2} \cdots \zeta_r^{n_r}$ in the r-dimensional space $\mathbf{C^r}$of complex lines $(\zeta_1, \zeta_2, \cdots, \zeta_r)$ with

$$|k; n_1, n_2, \cdots, n_r\rangle \longrightarrow C_{k;n_1,n_2,\cdots,n_r} \zeta_1^{n_1} \zeta_2^{n_2} \cdots \zeta_r^{n_r}, \tag{48}$$

a.e. The coefficients in (48) satisfy the recurrence formula

$$\sqrt{n_i} C_{k;n_1,n_2,\cdots,n_i,\cdots,n_r} = \sqrt{k-n} C_{k;n_1,n_2,\cdots,n_i-1,\cdots,n_r} \tag{49}$$

where $n = n_1 + n_2 + \cdots + n_r$. The solution, for all $i = 1, 2, \cdots, r$, is given by

$$C_{k;n_1,n_2,\cdots,n_r} = \left[\frac{(k-1)!}{n_1! n_2! \cdots n_r!(k-1-r)}\right]^{\frac{1}{2}}. \tag{50}$$

The creation generators a_i^+ act in this realization as

$$a_i^- \longrightarrow (k-1)\zeta_i - \zeta_i \sum_{j=1}^{r} \zeta_j \frac{\partial}{\partial \zeta_i}; \tag{51}$$

i.e; first order differential operators.

As in the previous cases, there exists a measure $\sigma(k; \zeta_1, \zeta_2, \cdots, \zeta_r)$ by means of which one can

define the inner product between two arbitrary functions. To compute this measure, we use the orthogonality of the Fock states $|k; n_1, n_2, \cdots, n_r\rangle$ which gives

$$\int\int\cdots\int d^2\zeta_1 d^2\zeta_2\cdots d^2\zeta_r \sigma(k;\zeta_1,\zeta_2,\cdots,\zeta_r) C_{k;n_1,n_2,\cdots,n_r} C_{k;n'_1,n'_2,\cdots,n'_r} \zeta_1^{n_1}\zeta_1^{n'_1}\zeta_2^{n_2}\zeta_2^{n'_2}\cdots\zeta_r^{n_r}\zeta_r^{n'_r}$$

$$= \delta_{n_1,n'_1}\delta_{n_2,n'_2}\cdots\delta_{n_r,n'_r} \tag{52}$$

Setting $\zeta_i = |\zeta_i|e^{i\theta}$ and assuming the isotropy of the measure, the relation (52) becomes

$$\int_0^\infty\int_0^\infty\cdots\int_0^\infty dx_1 dx_2\cdots dx_r \mu(k,x_1,x_2,\cdots,x_r) x_1^{n_1}x_2^{n_2}\cdots x_r^{n_r}$$

$$= \frac{n_1!n_2!\cdots n_r!(k-1-n)!}{(k-1)!} \tag{53}$$

where $\mu \equiv \pi^r\sigma$ and $x_i = |\zeta_i|^2$. Using the Mellin inverse transform [32], one obtain

$$\mu(k,x_1,x_2,\cdots,x_r) = \frac{(k-1+r)!}{(k-1)!}(1+x_1+x_2+\cdots+x_r)^{-(k+r)}. \tag{54}$$

Any function $f(\zeta_1,\zeta_2,\cdots,\zeta_r)$ can be written in the following form

$$f(\zeta_1,\zeta_2,\cdots,\zeta_r) = \mathcal{N}\langle k;\zeta_1^*,\zeta_2^*,\cdots,\zeta_r^*|f\rangle \tag{55}$$

where $|f\rangle$ is a generic element of the Fock space and the normalization constant is given by

$$\mathcal{N}(|\zeta_1|^2,|\zeta_2|^2,\cdots,|\zeta_r|^2) = (1+|\zeta_1|^2+|\zeta_2|^2+\cdots+|\zeta_r|^2)^{-\frac{k-1}{2}}. \tag{56}$$

It is interesting to note that the states $|k;\zeta_1,\zeta_2,\cdots,\zeta_r\rangle$ are nothing but the coherent states parameterizing the complex projective space $\mathbf{CP^r}$. They were used in the description of quantum Hall systems in higher dimension complex projective spaces [34]. In this respect, we believe that the generalized quantum A_r statistics can be linked to this subject.

5 Robertson-Schrödinger uncertainty relation

The main aim of this section is to show that the coherent states, derived in the previous section, minimize the Robertson-Schrödinger uncertainty relation [23-24]. The states minimizing this relation are called minimum uncertainty states (or intelligent states) [25-26]. For this end, we recall that for $2r$ observables (hermitian operators) $(X_1, X_2, \cdots, X_{2r}) \equiv X$, Robertson established the following uncertainty relation for the matrix dispersion σ

$$\det\sigma(X) \geq \det C(X) \tag{57}$$

where $\sigma_{\alpha\beta} = \frac{1}{2}\langle X_\alpha X_\beta + X_\beta X_\alpha\rangle - \langle X_\alpha X_\beta\rangle$, $(\alpha = 1,2,\cdots,2r)$, and C is the antisymmetric matrix of the mean commutators $C_{\alpha\beta} = -\frac{i}{2}[X_\alpha, X_\beta]$. Here $\langle O\rangle$ stands for the mean value of the operator O in a given quantum state which is generally a mixed state. For $r = 1$, inequality (57) coincides with Schrödinger uncertainty relation which gives the Heisenberg uncertainty relation when the term σ_{12} is vanishing.

5.1 Gazeau-Klauder coherent states

To show that the Gazeau-Klauder coherent states (32) minimize the uncertainty relation (57),i.e. $\det\sigma(X) = \det C(X)$, let define the hermitian operators $(X_1, X_2, \cdots, X_{2r})$ as

$$X_i = \frac{1}{2}(a_i^+ + a_i^-) \qquad X_{i+r} = \frac{i}{2}(a_i^+ - a_i^-) \tag{58}$$

in terms of the creation and annihilation operators of the quantum system described by the Hamiltonian H .

The matrix $A \equiv (a_1^-, a_2^-, \cdots, a_r^-, a_1^+, a_2^+, \cdots, a_r^+)$ is related to X as $X = UA$

$$U = \frac{1}{2}\begin{pmatrix} \mathbf{1}_r & \mathbf{1}_r \\ -i\mathbf{1}_r & i\mathbf{1}_r \end{pmatrix}$$

where $\mathbf{1}_r$ is $r \times r$ unit matrix. It follows that both matrices $\sigma(X)$ and $C(X)$ can be expressed in terms of matrices $\sigma(A)$ and $C(A)$:

$$\sigma(X) = U\sigma(A)U^T \qquad C(X) = UC(A)U^T \tag{59}$$

The eigenvalue equations $a_i^-|\omega_1, \omega_2, \cdots, \omega_r\rangle = \omega_i|\omega_1, \omega_2, \cdots, \omega_r\rangle$, provide us with the following relations between the matrix elements of $\sigma(A)$ and $C(A)$

$$\sigma_{ij} = 0 \qquad C_{ij} = 0 \tag{60}$$

$$\sigma_{i+r,j+r} = 0 \qquad C_{i+r,j+r} = 0 \tag{61}$$

$$\sigma_{i,j+r} = iC_{i,j+r} \qquad \sigma_{i+r,j} = -iC_{i+r,j}. \tag{62}$$

The last relations give $\det\sigma(A) = \det C(A)$ which in view of (59) (and the nondegeneracy of U, $\det U = (\frac{i}{2})^r$) leads to the needed equality in the Robertson-Schrödinger uncertainty relation (57), namely $\det\sigma(X) = \det C(X)$.

5.2 Klauder-Perelomov coherent states

Now, it remains to show that the Klauder-Perelomov, for bosonic and fermionic A_r statistics, minimize the Robertson-Schrödinger uncertainty relation. We first write the coherent states (45) and (55) as resulting from the action of some displacement operator on the lowest weight state (the vacuum). Indeed, by a more or less complicated calculus, one can show that coherent states (45) coincide with the vectors

$$\mathcal{D}(\eta_1, \eta_2, \cdots, \eta_r)|0, 0, \cdots, 0\rangle = \mathcal{D}(\eta_r)\cdots\mathcal{D}(\eta_2)\mathcal{D}(\eta_1)|0, 0, \cdots, 0\rangle \tag{63}$$

where the displacement operators $\mathcal{D}(\eta_i)$ are defined by

$$\mathcal{D}(\eta_1) = \exp(\eta_1 a_1^+ - \overline{\eta}_1 a_1^-) \quad , \quad \mathcal{D}(\eta_i) = \exp(\eta_i[a_{i-1}^-, a_i^+] - \overline{\eta}_i[a_i^-, a_{i-1}^+]) \tag{64}$$

for $i = 2, 3, \cdots, r$. The complex parameters occurring in the equations (63) and (64) are given in terms of variables labeling the coherent states (45) as $\tanh^2|\eta_1| = |z_1|^2 + |z_2|^2 + \cdots + |z_r|^2$,

$\tan^2|\eta_i| = |z_{i-1}|^{-2}(|z_i|^2 + |z_{i+1}|^2 + \cdots + |z_r|^2)$ for $i = 2, 3, \cdots, r$ and $\frac{\eta_i}{|\eta_i|} = \frac{z_i}{|z_i|}$.
Similarly, the coherent states obtained for fermionic A_r statistics derived from the expression (55) can be written as

$$\mathcal{D}(\eta'_1, \eta'_2, \cdots, \eta'_r)|0, 0, \cdots, 0\rangle = \mathcal{D}(\eta'_r)\cdots\mathcal{D}(\eta'_2)\mathcal{D}(\eta'_1)|0, 0, \cdots, 0\rangle \tag{65}$$

where the unitary operators $\mathcal{D}(\eta'_i)$ are defined as follows

$$\mathcal{D}(\eta'_1) = \exp(\eta'_1 a_1^+ - \overline{\eta'}_1 a_1^-) \quad , \quad \mathcal{D}(\eta'_i) = \exp(\eta'_i[a_i^+, a_{i-1}^-] - \overline{\eta'}_i[a_{i-1}^+, a_i^-]). \tag{66}$$

The complex variables ζ_i, labeling the fermionic A_r statistics states, are related to ones, parameterizing the displacement operators (66), as follows $\zeta_i = Z_1 Z_2 \cdots Z_i$, $(i = 1, 2, \cdots, r)$ with $Z_j = \frac{\eta'_j}{|\eta'_j|}\tan|\eta'_j|\cos|\eta'_{j+1}|$ for $j = 1, 2, \cdots, r-1$ and $Z_r = \frac{\eta'_r}{|\eta'_r|}\tan|\eta'_r|$.
To prove that the states (63) and (65) minimize the Robertson-Schrödinger uncertainty relation, we shall show that they are eigenstates of the linear combination of Jacobson generators $A_i^- \equiv A_i^-(u, v) = u_{ij}a_j^- + v_{ij}a_j^+$ (summation over repeated indices). To simplify our notations, we denote by $|coh, s = \pm 1\rangle$ the coherent states for bosonic $(s = 1)$ and fermionic $(s = -1)$ A_r statistics. Using the triple relation commutation, one get

$$\mathcal{D}^\dagger a_i^+ \mathcal{D} = x_{ij}a_j^- + y_{ij}a_j^+ + z_{ijk}[a_j^-, a_k^+] \tag{67}$$

where $\mathcal{D}$ is given by (64) (resp. (66)) for bosonic A_r statistics (resp. fermionic A_r statistics). The complex parameters x_{ij}, y_{ij} and z_{ijk} are functions of the variables labeling the coherent states (The expressions of x_{ij}, y_{ij} and z_{ijk} can be obtained by using the trilinear relations (3) and (4) coupled with Baker-Campbell-Hausdorff relation). From (67), one obtain

$$\mathcal{D}^\dagger A_i^- \mathcal{D}|0, 0, \ldots, 0\rangle =$$
$$\Big[(u_{ij}x_{jk} + v_{ij}y^*_{jk})a_k^- + (u_{ij}y_{jk} + v_{ij}x^*_{jk})a_j^+ + (u_{ij}z_{jkl} + v_{ij}z^*_{jkl})[a_k^-, a_l^+]\Big]|0, 0, \ldots, 0\rangle$$

Since $a_j^-|0, 0, \ldots, 0\rangle = 0$ and $[a_k^-, a_l^+]|0, 0, \ldots, 0\rangle = (k + \frac{s-1}{2})\delta_{kl}|0, 0, \ldots, 0\rangle$, the coherent states (63) and (65) are eigenstates of A_i^- if the $r \times r$ matrices u, v, x and y satisfy the condition $uy + vx^* = 0$ and we have

$$A_i^-|coh, s = \pm 1\rangle = (k + \frac{s-1}{2})\sum_{jl}(u_{ij}z_{jll} + v_{ij}z^*_{jll})|coh, s = \pm 1\rangle \tag{68}$$

In the last step in our proof, we consider the quadrature components X_i and X_{i+r}, defined previously, which can be related to operators $A \equiv (A_1^-, A_2^-, \cdots, A_r^-, A_1^+, A_2^+, \cdots, A_r^+)$ as follows

$$X = U\Omega^{-1}A \tag{69}$$

where the matrix Ω (assumed to be invertible) is defined as

$$\Omega = \begin{pmatrix} u & v \\ v^* & u^* \end{pmatrix}$$

and the matrix U is given above. Using the transformation (69) one can verify easily the following expressions of dispersion and covariance matrices $\sigma(X)$ and $C(X)$

$$\sigma(X) = (U\Omega^{-1})\sigma(A)(U\Omega^{-1})^T \quad , \quad C(X) = (U\Omega^{-1})C(A)(U\Omega^{-1})^T \tag{70}$$

in terms of the dispersion and covariance of the A's operators. According the eigenvalues equations (68), the matrix elements of $\sigma(A)$ and $C(A)$ are related by relations similar to ones given by (60), (61) and (62). Then, one has $\det \sigma(A) = \det C(A)$ which implies $\det \sigma(X) = \det C(X)$. Finally, we conclude that the Klauder-Perelomov coherent states, arising from the Bargmann realizations of bosonic and fermionic A_r statistics, minimize the Robertson-Shrödinger uncertainty relation and they are, in this respect, intelligent.

6 Conclusion

This paper was devoted to the generalized A_r statistics. We have studied the associated Fock representations. We have obtained the Fock spaces associated to bosonic ($s = 1$) and fermionic ($s = -1$) A_r statistics. In the limit $k \longrightarrow \infty$ (k index labeling the irreducible Fock representations), bosonic as well as fermionic A_r statistics reduce to the standard Bose statistics. The A_r statistics system becomes a collection of ordinary bosons and the Jacobson generators coincide with creation and annihilation operators of conventional degrees of freedom. We have developed the Bargmann realizations of the Fock spaces and determined the differential actions of the Jacobson generators. We have shown that the so-called Klauder-Perelomov and Gazeau-Klauder coherent states emerge, in a natural way, in these realizations. The measures, by means of which we define the inner product of two analytical functions for each considered realization, are computed. They turn out to be the measures with respect which the coherent states constitute over-complete sets. We point out that the existence of two distinct Bargmann representations, studied in sections 3 and 4, arises from choosing either the creation or the annihilation operator having a simple form similar to the ordinary Bose case; indeed in the latter the two coincide and there is only one Bargmann realisation, but they are necessarily distinct in the case under discussion. We shown also that all obtained coherent states are intelligent. In other words, the states give the minimum of the Robertson-Schrödinger uncertainty relation. As first continuation, it would be interesting to study a complete classification of intelligent states associated with A_r statistics. Furthermore, the results and tools presented in this article can be extended to quantum statistics associated with other classical Lie algebras and super-algebras. Finally, we believe that the generalized A_r statistics can be applied in the study of quantum Hall effect in higher dimension spaces [34,35]. We hope to report on this subject in a forthcoming work.

Acknowledgements

Thanks are due to the referees for pertinent and constructives remarks.

References

[**1**] H.S. Green, Phys. Rev. **90** (1953) 270.

[**2**] O.W. Greenberg and A.M.L. Messiah, Phys. Rev. B **138** (1965) 1155;
J. Math. Phys. **6** (1965) 500.

[**3**] Y. Ohnuki and S. Kamefuchi, *Quantum field theory and parastatistics*
(Springer, Berlin, 1982).

[**4**] J.M. Leinaas and J. Myrheim, Nuovo Cimento B **37** (1977) 1.

[**5**] F. Wilczek, Phys. Rev. Lett. **49** (1982) 957.

[**6**] T.D. Palev, *Lie algebraic aspects of quantum statistics. Unitary quantization (A-quantization)*, Preprint JINR E17-10550 (1977) and hep-th/9705032.

[**7**] R.B. Laughlin, Phys. Rev. Lett. **50** (1983) 1395.

[**8**] B.I. Halperin, Phys. Rev. Lett. **52** (1984) 1583.

[**9**] R.B. Laughlin, Phys. Rev. Lett. **60** (1988) 2677.

[**10**] A. Strominger, Phys. Rev. Lett. **71** (1993) 3397.

[**11**] O.W. Greenberg , Phys. Rev. Lett. **64** (1990) 705; Phys. Rev. D **43**(1991) 4111.

[**12**] M. Daoud and M. Kibler, Phys. Lett. A **206** (1995) 13; M.R. Kibler, J. Meyer and M. Daoud, in *Symmetry and Structutral Properties of Condensed Matter*, eds T. Lulek, W. Florek and B. Lulek (World Scientific, Singapore, 1997).

[**13**] F.D.M. Haldane, Phys. Rev. Lett. **67** (1991) 937.

[**14**] S. Kamefuchi and Y. Takahashi, Nucl. Phys. **36** (1962) 177; C. Ryan and E.C.G. Sudarshan, Nucl. Phys. **47** (1963) 207.

[**15**] V.G Kac, Adv. Math. **26** (1977) 8.

[**16**] T.D. Palev, Czech. J. Phys. **B 29** (1979) 91; Rep. Math. Phys. **18** (1980) 177; Rep. Math. Phys. **18** (1980) 129; J. Math. Phys **21** (1980) 1293.

[**17**] N.I. Stoilova and J. Van der Jeugt, preprint math-phys/0409002.

[**18**] N. Jacobson, Amer. J. Math. **71** (1949) 149.

[**19**] M. Arik and D.D. Coon J. Math. Phys. **17** (1976) 524.

[**20**] M. Daoud and M. Kibler, Phys. Part. Nuclei (Suppl 1) **33** (2002) S43; Phys. Lett. A **321** (2004) 147 .

[**21**] J. K. Sharma, C.L. Mehta and E.C.G. Sudarshan, J. Math. Phy. **19** (1978) 2089.
J. K. Sharma, C.L. Mehta, N. Mukunda and E.C.G. Sudarshan, J. Math. Phy. **22** (1981) 78.

[**22**] S. Jing and C.A. Nelson, preprint hep-th/9807048 .

[**23**] H.P. Robertson, Phys. Rev. **35** (1930) 667A, Phys. Rev. **46** (1934) 794.

[**24**] E.Schrödinger, Sitzungsber. Preuss. Acad. Wiss. Phys-Math. Klasse, **19** (Berlin,1930) 269.

[**25**] V.V. Dodonov, E. Kurmyshev and V.I. Man'ko, Phys. Lett.A **79** (1980) 150.

[**26**] D.A. Trifonov, J. Phys. A: Mathematical and general **30** (1997) 5941; Phys. Lett. A **48** (1974) 165; J. Math. Phy. **35** (1994) 2297.

[**27**] J.R. Klauder, J. Phys. A: Math. Gen.**29** (1996) L 293; J-P. Gazeau and J-R. Klauder, J. Phys. A: Math. Gen.**32** (1999) 123.

[**28**] J.R. Klauder and B.S Skagerstam, *Coherent states-Applications in Physics and Mathematical Physics* (World Scientific,Singapore,1985).

[**29**] A. Perelomov, *Generalized Coherent States and their Applications*, Texts and Monographs in Physics, (Spinger-Verlag,1986).

[**30**] B. Basu-Mallick, preprint cond-mat/0201074.

[**31**] A.P. Polychronakos, in *Les Houches Lectures* (1998), preprint hep-th/9902157.

[**32**] H.Bateman, Table of integral transforms, Vol $\underline{1}$ (1954), Ed A.Erdélyi (New York, Mc Graw Hill) .

[**33**] K. Fujii and K. Funahashi, preprint quant-phys/9704011.

[**34**] D. Karabali and V.P. Nair, Nucl. Phys. B **641**(2002) 533; Nucl. Phys. B **679** (2004) 427.

[**35**] A. Jellal, Nucl. Phys. B **725** (2005) 554.